Derivation of the mean radial displacement, $\bar{l}_\rho^{\rm 2D}$, in 2 dimensions

Markus Dittrich

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1 Introduction

In the following we derive an expression for the mean radial displacement, $\bar{l}_\rho^{\rm 2D}$, for diffusing 2D molecules in analogy to the well known expression for $\bar{l}_r$ in three dimensions.

2 Derivation

Starting point is the solution of the diffusion equation in 2 dimensions

$$\frac{\partial c(\rho,t)}{\partial t} = D\nabla^2 c(\rho,t) \tag{1}$$

which, for a point source of M molecules released at the origin at time $t = 0$ can be shown to be

$$c(\rho,t) = \frac{M}{4\pi Dt} e^{-\frac{\rho^2}{4Dt}} \tag{2}$$

(see also Eqs. 3.1, 3.2 in [1]). Therefore, the probability density for a molecule being in a radial shell of thickness $d\rho$ around the origin is given by

$$p^{\rm 2D}(\rho,t) = \frac{(2\pi\rho)d\rho}{4\pi Dt} e^{-\frac{\rho^2}{4Dt}} \tag{3}$$

or, in normalized coordinates $\tilde{\rho} = \frac{\rho}{\lambda}$ with the normalization constant $\lambda = \sqrt{4Dt}$

$$p^{\rm 2D}(\tilde{\rho},t) = 2e^{-\tilde{\rho}^2}(\tilde{\rho}d\tilde{\rho}) \quad . \tag{4}$$

Hence, the mean radial displacement of a 2D molecule is given by

$$\bar{l}_\rho^{\rm 2D} = 2\lambda \int_0^\infty \tilde{\rho}^2 e^{-\tilde{\rho}^2} d\tilde{\rho} \tag{5}$$

Since

$$\begin{aligned} \int_0^\infty t^{2n} e^{-at^2} dt &= \frac{\Gamma(n+\frac{1}{2})}{2a^{n+\frac{1}{2}}} \\ \Rightarrow \int_0^\infty \tilde{\rho}^2 e^{-\tilde{\rho}^2} d\tilde{\rho} &= \frac{\Gamma(\frac{3}{2})}{2} = \frac{\sqrt{\pi}}{4} \end{aligned} \tag{6}$$

and from this

$$\boxed{\bar{l}_{\rho}^{\mathrm{2D}} = 2\lambda\frac{\sqrt{\pi}}{4} = \sqrt{\pi D t}} \tag{7}$$

References

[1] R. Kerr, T.M. Bartol, B. Kaminsky, M. Dittrich, J.C.J. Chang, S. Baden, T.J. Sejnowski, and J.R. Stiles, Fast Monte Carlo Simulation Methods for Biological Reaction-Diffusion Systems in Solution and on Surfaces (2008), SIAM J. Sci. Comput., 30:3126-3149.