

Reactions with More than Two Components

Three-way reactions in MCell

The rate of reaction of a molecule that can engage in a three-way reaction with reactants I and J at concentrations ρ_I and ρ_J is $\kappa\rho_I\rho_J$. Suppose that a single molecule moves a distance R while sweeping out an interaction area A . Then the expected number of hits, assuming that the concentration of I and J is low, is

$$n_{\text{hits}} = RA\rho_I \cdot RA\rho_J$$

Thus, the expected number of hits for a molecule with a diffusion length constant of λ is

$$n_{\text{hits}} = \int_0^\infty \rho_I\rho_JA^2R^2 \frac{4\pi R^2}{\pi^{3/2}\lambda^3} e^{-R^2/\lambda^2} dR = \frac{3}{2}\rho_I\rho_JA^2\lambda^2$$

If we let p be the probability of reaction, then

$$\kappa\rho_I\rho_J\Delta t = p \cdot n = p \cdot \frac{3}{2}\rho_I\rho_JA^2\lambda^2$$

Solving for p gives

$$p = \frac{\kappa}{6DA^2}$$

assuming that Δt is the time step for the moving molecule. If we let all three reactants move and react—let us number them 1, 2, and 3—then we matching the total rate gives

$$\kappa\rho_1\rho_2\rho_3\Delta t = \frac{3}{2}\rho_1\rho_2\rho_3A^2 \left(p_1 \frac{\Delta t}{\Delta t_1} \lambda_1^2 + p_2 \frac{\Delta t}{\Delta t_2} \lambda_2^2 + p_3 \frac{\Delta t}{\Delta t_3} \lambda_3^2 \right)$$

where in general the individual molecules may move with custom timesteps Δt_i . We let $p_1 = p_2 = p_3 = p$ to give

$$p = \frac{\kappa}{6(D_1 + D_2 + D_3)A^2}$$

This solution also works for the cases where some of the reactants can't move (as D_i will be zero and will drop out of the equation).

Now suppose that the reaction takes place near a surface such that for a fraction a of the distance, the molecule sweeps out $A^* < A$ of area instead of A . The expected number of hits is then

$$n_{\text{hits}}^* = R\rho_I ((1-a)A + aA^*) R\rho_J ((1-a)A + aA^*)$$

which we can rewrite as

$$n_{\text{hits}}^* = n_{\text{hits}} \left((1-a)^2 + 2a(1-a)\frac{A^*}{A} + a^2 \left(\frac{A^*}{A} \right)^2 \right)$$

where the first term occurs when both hits are in the unconstrained space, the second when one target molecule is in the unconstrained space and one is in the constrained space, and the third when both targets are in the constrained space. If we multiply the probability of reaction by the

inverse of the fractional areas for each target, i.e., by A/A^* if one target is in the constrained space and $(A/A^*)^2$ when both are in the constrained space, we then find that the total rate of reaction is

$$pn_{\text{hits}} \left((1-a)^2 \cdot 1 \cdot 1 + 2a(1-a) \frac{A^*}{A} \cdot 1 \cdot \frac{A}{A^*} + a^2 \left(\frac{A^*}{A} \right)^2 \cdot \frac{A}{A^*} \cdot \frac{A}{A^*} \right) = pn_{\text{hits}}$$

That is, the reaction rate is unchanged, which is exactly what we want. Since a is arbitrary, we can make a differentially small and thus the result holds for arbitrary restrictions of the swept area.

3-way reactions plus a surface in MCell

If we have three reactants but one of them is on a surface, we can calculate the probability of the moving molecule both striking the surface and hitting the other reactants. If the molecule is a distance h above the surface, then the probability of hitting the surface is

$$p_{\text{surfhit}} = \frac{1}{2} \int_h^\infty dr_z \cdot \frac{1}{\pi^{1/2}\lambda} e^{-r_z^2/\lambda^2}$$

But the molecule also diffuses in the xy plane, and this determines the length of the collision cylinder. If the diffusion distance is r_{xy} in that plane, then the total diffusion length is $r = \sqrt{r_{xy}^2 + r_z^2}$ and the swept volume is $A\sqrt{r_{xy}^2 + r_z^2}$. For a given r_z , then, the probability of hitting another free molecule is

$$p_{\text{freehit}} = \int_0^\infty dr_{xy} \cdot \frac{2\pi r_{xy}}{\pi\lambda^2} e^{-r_{xy}^2/\lambda^2} \cdot \left(A\sqrt{r_{xy}^2 + r_z^2} \right) \cdot \rho_1$$

where ρ_1 is the density of the other free molecule. Thus, the probability of hitting the surface from distance h and also hitting a diffusing partner is

$$\begin{aligned} p_{\text{hit}}(h) &= \frac{1}{2} \int_h^\infty dr_z \frac{1}{\pi^{1/2}\lambda} e^{-r_z^2/\lambda^2} \int_0^\infty dr_{xy} \cdot \frac{2\rho_1 A}{\lambda^2} r_{xy} \sqrt{r_{xy}^2 + r_z^2} \cdot e^{-r_{xy}^2/\lambda^2} \\ &= \frac{\rho_1 A}{\pi^{1/2}\lambda^3} \int_h^\infty dr_z \cdot e^{-r_z^2/\lambda^2} \int_0^\infty dr_{xy} \cdot r_{xy} \sqrt{r_{xy}^2 + r_z^2} e^{-r_{xy}^2/\lambda^2} \end{aligned}$$

The inner integral evaluates to $\frac{\lambda^2}{2} r_z + \frac{\pi^{1/2}\lambda^3}{4} e^{r_z^2/\lambda^2} \text{erfc}\left(\frac{r_z}{\lambda}\right)$, giving

$$p_{\text{hit}}(h) = \frac{\rho_1 A}{2\pi^{1/2}\lambda} \int_h^\infty dr_z \cdot \left(r_z e^{-r_z^2/\lambda^2} + \frac{\pi^{1/2}\lambda}{2} \text{erfc}\left(\frac{r_z}{\lambda}\right) \right)$$

The outer integral works out to $\lambda^2 e^{-h^2/\lambda^2} - \frac{\lambda h \pi^{1/2}}{2} \text{erfc}\left(\frac{h}{\lambda}\right)$:

$$p_{\text{hit}}(h) = \frac{\rho_1 A}{2\pi^{1/2}} \left(\lambda e^{-h^2/\lambda^2} - \frac{h \pi^{1/2}}{2} \text{erfc}\left(\frac{h}{\lambda}\right) \right)$$

which we can integrate over the entire column above a surface molecule of area B to get the expected number of hits:

$$n_{\text{hit}} = \int_0^\infty dh \cdot B \cdot \rho_2 \cdot p_{\text{hit}}(h) = \frac{\rho_1 A \rho_2 B}{2\pi^{1/2}} \cdot \frac{3\lambda^2 \pi^{1/2}}{8} = \frac{3}{16} \lambda^2 \rho_1 \rho_2 A B$$

Since $n_{\text{hit}} \cdot p_{\text{rx}}$ should be equal to the bulk reaction rate $\kappa \rho_1 \rho_2 \Delta t$, we have

$$p_{\text{rx}} = \frac{4\kappa}{3DAB}$$

if only species 2 diffuses (note that $\lambda^2/\Delta t = 4D$). If both volume molecules diffuse, the total number of hits in a time Δt is

$$\frac{\Delta t}{\Delta t_1} n_{\text{hit},1} + \frac{\Delta t}{\Delta t_2} n_{\text{hit},2} = \frac{3}{16} (4D_1 + 4D_2) \rho_1 \rho_2 AB \Delta t$$

so that

$$p_{\text{rx}} = \frac{4\kappa}{3AB(D_1 + D_2)}$$

If the surface can be hit from either side, the number of hits doubles, so the reaction rate should be halved:

$$p'_{\text{rx}} = \frac{2\kappa}{3AB(D_1 + D_2)}$$

Three-way reactions with two surface components

The standard computation for the number of hits against a single surface is

$$n_{\text{hit}} = \frac{\rho_1 A \lambda}{2\sqrt{\pi}}$$

and the probability that a grid element is filled with an appropriate surface molecule is $\sigma_i A$ so that if we demand that we strike the first surface molecule directly and the second is adjacent, then the productive hit rate is (keeping in mind that either of the two surface molecules can be the initial target)

$$n_{\text{productive}} = \frac{3\rho_1 \sigma_2 \sigma_3 A^3 \lambda}{\sqrt{\pi}}$$

The desired number of reactions is $\kappa \rho_1 \sigma_2 \sigma_3 A \Delta t$ (if appropriate units are used for the rate constant), so that

$$p_{\text{rx}} = \kappa \frac{\sqrt{\pi}}{3A^2 v}$$

In contrast, if one of the components is the surface itself (let's assign it to σ_3) and the other is the molecule, the number of productive collisions drops by a factor of six (no neighbors, and only one target), $\sigma_3 A = 1$ in the probability calculation and σ_3 is typically omitted entirely from the bulk equation, giving

$$p'_{\text{rx}} = \kappa \frac{2\sqrt{\pi}}{Av}$$

Three-way reactions with all surface components

If all components are in the surface, the total number of reactions per timestep for a single molecule should be $\kappa \sigma_2 \sigma_3 \Delta t$ and the actual probability of finding the appropriate neighbors is $3\sigma_2 A \cdot 2\sigma_3 A$. Thus the reaction probability should be

$$p_{\text{rx}} = \frac{\kappa}{6A^2} \Delta t$$

N-way reactions in MCell

Generalizing to $N + 1$ reactants (one moving and N targets), where N is a positive integer, we find that the collision rate is

$$n_{\text{hits}} = \int_0^\infty dR \cdot \prod_{i=1}^N \rho_i \cdot (RA)^N \cdot \frac{4\pi R^2}{\pi^{3/2}\lambda^3} e^{-R^2/\lambda^2} = \prod_{i=1}^N \rho_i \cdot \frac{2\lambda^N A^N}{\sqrt{\pi}} \Gamma\left(\frac{N+3}{2}\right)$$

and the bulk rate is $\kappa \cdot \prod_{i=1}^N \rho_i \cdot \Delta t$, so equating probabilities gives

$$p = \frac{\kappa \sqrt{\pi} \Delta t}{2\lambda^N A^N \Gamma\left(\frac{N+3}{2}\right)}$$

Note that $\Gamma(N) = (N-1)!$ and $\Gamma(N + \frac{1}{2}) = \sqrt{\pi} \cdot 2^{-2N} \cdot (2N)!/N! = \sqrt{\pi} \cdot \prod_{i=1}^N \frac{2i-1}{2}$. If we have multiple moving molecules,

$$\kappa \cdot \prod_{i=1}^{N+1} \rho_i \cdot \Delta t = \prod_{i=1}^{N+1} \rho_i \cdot \frac{2A^N}{\sqrt{\pi}} \Gamma\left(\frac{N+3}{2}\right) \cdot \sum_{i=1}^{N+1} p_i \frac{\Delta t}{\Delta t_i} \lambda_i^N$$

so that, if we set all the p_i to be equal,

$$p = \frac{\kappa \sqrt{\pi}}{2\Gamma\left(\frac{N+3}{2}\right) A^N \sum \frac{\lambda_i^N}{\Delta t_i}}$$

By induction on the result for pairs of targets in the 3-way case, we also see that if a target i is hit in a restricted space, the reaction probability should be multiplied by A/A_i^* .

Higher order reactions with single surfaces

The primary equation for $p_{\text{hit}}(h)$ remains the same for higher-order reactions except that $\rho_1 A r$ turns into $\prod_{i=1}^{N-1} \rho_i A r$ (recall that $r = \sqrt{r_{xy}^2 + r_z^2}$). Otherwise the integration is the same. The general formulae for such integrals is rather tricky, but the first few values for n_{hit} are listed here:

$$\begin{aligned} n_{\text{hit}}^{(N=2)} &= \frac{3}{16} \lambda^2 A B \prod_{i=1}^N \rho_i \\ n_{\text{hit}}^{(N=3)} &= \frac{1}{2\sqrt{\pi}} \lambda^3 A^2 B \prod_{i=1}^N \rho_i \\ n_{\text{hit}}^{(N=4)} &= \frac{15}{32} \lambda^4 A^3 B \prod_{i=1}^N \rho_i \\ n_{\text{hit}}^{(N=5)} &= \frac{3}{2\sqrt{\pi}} \lambda^5 A^4 B \prod_{i=1}^N \rho_i \\ n_{\text{hit}}^{(N=6)} &= \frac{105}{64} \lambda^5 A^5 B \prod_{i=1}^N \rho_i \\ n_{\text{hit}}^{(N=7)} &= \frac{6}{\sqrt{\pi}} \lambda^6 A^5 B \prod_{i=1}^N \rho_i \end{aligned}$$

The author conjectures that the formula for even N is

$$\frac{(N+1)!}{4 \cdot 2^{3N/2} (N/2)!} \lambda^N A^{N-1} B \prod_{i=1}^N \rho_i$$

and for odd N is

$$\frac{((N+1)/2)!}{4\sqrt{\pi}} \lambda^N A^{N-1} B \prod_{i=1}^N \rho_i$$

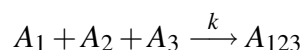
These have been checked up to $N = 12$ in Maple 10, but these formulae have not been proven.

Higher order reactions with multiple surface components

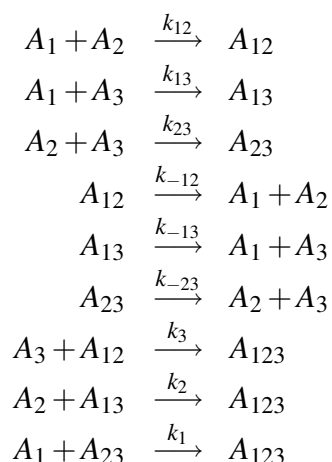
Each time one adds a molecular surface component, one adds a factor of $\frac{1}{A}$ the first time (if one is already hitting a surface and a molecule needs to be there also), $\frac{1}{3A}$ the second time, $\frac{1}{2A}$ the third time, and $\frac{1}{A}$ the fourth time. More than four molecules cannot be found by adjacent search; if this is generalized to a wider search, then if the extra partner can be found in one of n places, the reaction probability changes by $\frac{1}{nA}$. In addition, if there are k possible surface targets total, the reaction probability should be multiplied by an additional $\frac{1}{k}$.

Converting 3-way reactions to 2-way reactions

Suppose we have a three-way reaction



which we wish to approximate by nine bimolecular reactions:



At quasi-steady state, we want to match rate of entry in to the A_{123} state:

$$kA_1A_2A_3 = k_1A_1A_{23} + k_2A_2A_{13} + k_3A_3A_{12} \quad (1)$$

And we also wish to keep the quasi-steady state concentrations of the intermediates A_{12} , A_{13} , and A_{23} low compared to the starting materials. In general, we will have

$$\frac{d}{dt}A_{hi} = -k_jA_jA_{hi} - k_{-hi}A_{hi} + k_{hi}A_hA_i \approx 0$$

so that

$$A_{hi} \approx \frac{k_{hi}A_hA_i}{k_{-hi} + k_jA_j}$$

If we want this to be roughly independent of the concentration of A_j then we require $k_{-hi} \gg k_jA_j$ and can rewrite this as

$$A_{hi} \approx \frac{k_{hi}}{k_{-hi}} A_h A_i \left(1 - \frac{k_j A_j}{k_{-hi}} \right)$$

If we further require that A_{hi} be small compared to A_h and A_i , we also require $\frac{k_{hi}}{k_{-hi}} \ll \frac{1}{\max(A_h, A_i)}$. Let A_+ be the largest value of any of the A_i during a simulation. Furthermore, let us set all k_j to be k^* , all k_{hi} to be k^\ddagger and all k_{-hi} to be k^\ddagger . Then our constraints require that $k^\ddagger \gg k^* A_+$ and $k^\ddagger \ll k^\ddagger \frac{1}{A_+}$; taken together, $k^\ddagger \approx k^*$ is a valid solution, so we may as well make the two the same, k' . Thus, we have a forward reaction rate k' for all binding reactions and a backward reaction rate k^\ddagger for dissociation of the intermediates.

Thus, equation (1) becomes

$$kA_1A_2A_3 \approx k'A_1 \frac{k'}{k^\ddagger} A_2A_3 + k'A_2 \frac{k'}{k^\ddagger} A_1A_3 + k'A_3 \frac{k'}{k^\ddagger} A_1A_2 = 3 \frac{k'^2}{k^\ddagger} A_1A_2A_3$$

with a first-order error term

$$-\frac{k'^3}{k^\ddagger^2} A_1A_2A_3 (A_1 + A_2 + A_3)$$

If we let $k^\ddagger = \alpha k'$, where $\alpha \gg A_+$, we then have

$$k \approx \frac{3}{\alpha} k' - \frac{A_1 + A_2 + A_3}{\alpha^2} k'$$

Thus,

$$k_1 = k_2 = k_3 = k_{12} = k_{13} = k_{23} = k' \approx \frac{1}{3} \alpha k$$

and

$$k_{-12} = k_{-13} = k_{-23} = k^\ddagger \approx \frac{1}{3} \alpha^2 k$$

Note that our fractional error is approximately $1/\alpha$, i.e. if we let $\alpha = 100A_+$ our fractional error would be under 1%.

Notes on Units

When rates are measured for bimolecular reactions between a volume molecule and a surface molecule, one can lay down a surface with known (or measurable) area \mathcal{A} in a solution of volume \mathcal{V} . You then add n_1 volume molecules (concentration $\rho_1 = n_1/\mathcal{V}$ in units of #/unit volume) and n_2 surface molecules (at density $\sigma_2 = n_2/\mathcal{A}$) and measure

$$\frac{d\rho_1}{dt} = -k_\rho \rho_1 \sigma_2$$

where k_ρ is the rate constant with units of $\text{area} \cdot \#^{-1} \cdot \text{s}^{-1}$. One can equally well write this as

$$\frac{d\sigma_2}{dt} = -k_\sigma \rho_1 \sigma_2$$

where k_σ has units of $\text{volume} \cdot \#^{-1} \cdot \text{s}^{-1}$. Of course, the numbers of molecules reacting are the same, so that

$$-k_\sigma \mathcal{V}^{-1} n_1 n_2 = \mathcal{A} \frac{d\sigma_2}{dt} = \frac{dn_2}{dt} = \frac{dn_1}{dt} = \mathcal{V} \frac{d\rho_1}{dt} = -k_\rho \mathcal{A}^{-1} n_1 n_2$$

We can now let $k_n = k_\sigma \mathcal{V}^{-1} = k_\rho \mathcal{A}^{-1}$ and write

$$\frac{dn_\star}{dt} = -k_n n_1 n_2$$

But one can also define $\rho_2 = \sigma_2 \cdot \frac{\mathcal{A}}{\mathcal{V}}$, that is, treat the surface molecule as if it were a volume molecule, and then

$$\frac{d\rho_\star}{dt} = \frac{dn_\diamond}{dt} \mathcal{V}^{-1} = -k_n \rho_1 \rho_2 \mathcal{V}^{-1} = -k_\star \rho_1 \rho_2$$

where $k_\star = k_n \cdot \mathcal{V}^{-1}$; here \star stands for one of 1 or 2, while \diamond stands for the other.

If one is performing a stochastic calculation, the total number of hits on all surface molecules in a short time Δt is

$$n_2 \frac{\rho_1 A \lambda}{2\sqrt{\pi}} \Delta t$$

where A is the area of a single surface molecule. From the well-mixed continuum approximation, the probability should be scaled such that

$$-p_{\text{rx}} \cdot n_2 \frac{\rho_1 A \lambda}{2\sqrt{\pi}} = \frac{dn_\diamond}{dt} \Delta t = \mathcal{V} \frac{d\rho_\star}{dt} \Delta t = -k_\star n_2 \rho_1 \Delta t$$

so that $p_{\text{rx}} = k_\star \cdot 2\sqrt{\pi}/A\nu$ where $\nu = \lambda/\Delta t$. Thus, we can use the volumetric rate constant k_\star where we need only convert from molarity to #/unit volume—we need not know the original volume of the test sample or the area of membrane in it, as long as the value of k_\star is reported. Conveniently, one can measure k_\star without even knowing the area of the membrane.

However, if we add a second surface component at density σ_3 , the above is no longer true since the reaction rate is no longer proportional to the numbers of each molecule. In particular,

$$\frac{d\rho_1}{dt} = -k_\rho \rho_1 \sigma_2 \sigma_3$$

$$\frac{d\sigma_{i \in \{2,3\}}}{dt} = -k_\sigma \rho_1 \sigma_2 \sigma_3$$

defines the reactions, but now

$$\frac{dn_1}{dt} = \mathcal{V} \frac{d\rho_1}{dt} = -k_\rho n_1 n_2 n_3 \mathcal{A}^{-2}$$

$$\frac{dn_i}{dt} = \mathcal{A} \frac{d\sigma_i}{dt} = -k_\sigma n_1 n_2 n_3 \mathcal{A}^{-1} \mathcal{V}^{-1}$$

so that $k_n = k_\sigma \mathcal{A}^{-1} \mathcal{V}^{-1} = k_\rho \mathcal{A}^{-2}$. If we try the same trick of converting σ_2 and σ_3 to volumes, we find that -k

$$-k_\star \rho_1 \rho_2 \rho_3 = \frac{d\rho_\star}{dt} = \mathcal{V}^{-1} \frac{dn_\star}{dt} = -k_n n_1 n_2 n_3 \mathcal{V}^{-1} = -k_n \rho_1 \rho_2 \rho_3 \mathcal{V}^2$$

so that $k_n = k_\star \mathcal{V}^{-2}$. When we try to match total numbers of molecules reacting, we find that there are

$$p_{\text{rx}} \frac{3\rho_1 \sigma_2 \sigma_3 A^3 \lambda}{\sqrt{\pi}} \cdot \frac{\mathcal{A}}{A}$$

total reactions in time Δt in a stochastic treatment (the factor of \mathcal{A}/A arises from the difference between the per-receptor area and the total surface area), and

$$\frac{dn_{\star}}{dt} \Delta t = \mathcal{V} \frac{d\rho_{\star}}{dt} \Delta t = -k_{\star} \rho_1 \rho_2 \rho_3 \mathcal{V} \Delta t = -k_{\star} \rho_1 \sigma_2 \sigma_3 \frac{\mathcal{A}^2}{\mathcal{V}} \Delta t$$

from the deterministic continuum equations. Equating the two (with the correct sign) gives

$$p_{rx} = k_{\star} \frac{\sqrt{\pi}}{3A^2 \mathcal{V}} \cdot \frac{\mathcal{A}}{\mathcal{V}}$$

This is problematic because the probability of reaction now depends on the surface to volume ratio \mathcal{A}/\mathcal{V} ; what we need is $k_{\star} \frac{\mathcal{A}}{\mathcal{V}}$. Fortunately, $k_{\sigma} = k_n \mathcal{A} \mathcal{V} = k_{\star} \frac{\mathcal{A}}{\mathcal{V}}$. Thus, the only appropriate rate constant for three-molecule reactions is k_{σ} , which has units of volume \cdot area \cdot # $^{-2} \cdot$ s $^{-1}$.